SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE

READ INSTRUCTIONS BEFORE COMPLETING FORM

1. REPORT NUMBER: GWU/IMSE/Serial T-485/84	2. GOVT ACCESSION NO.
3. RECIPIENT'S CATALOG NUMBER	
4. TITLE (and Subtitle): SOLUTION AND SENSITIVITY ANALYSIS OF RESDYN MODEL 1a USING SENSUMT	5. TYPE OF REPORT & PERIOD COVERED: SCIENTIFIC
	6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s): ANTHONY V. FIACCO FRANK A. AMODEO	8. CONTRACT OR GRANT NUMBER(s): N00014-83-K-0215
9. PERFORMING ORGANIZATION NAME AND ADDRESS: GEORGE WASHINGTON UNIVERSITY INSTITUTE FOR MANAGEMENT SCIENCE AND ENGINEERING WASHINGTON, DC 20052	10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS
11. CONTROLLING OFFICE NAME AND ADDRESS: OFFICE OF NAVAL RESEARCH ARLINGTON, VIRGINIA 22217	12. REPORT DATE: 30 March 1984
	13. NUMBER OF PAGES: 20
14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office)	15. SECURITY CLASS. (of this report): UNCLASSIFIED
	15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

APPROVED FOR PUBLIC SALE AND RELEASE; DISTRIBUTION UNLIMITED.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

RESOURCE ALLOCATION
DYNAMIC ALLOCATION
MAINTENANCE
NONLINEAR PROGRAMMING
PARAMETERS
SENSITIVITY ANALYSIS

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

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S/N 0102-LF-014-6601

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SOLUTION AND SENSITIVITY ANALYSIS OF RESDYN MODEL 1a USING SENSUMT

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Anthony V. Fiacco
Frank A. Amodeo

GWU/IMSE/Serial T-485/84
30 March 1984

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Research Supported by
Office of Naval Research
Contract N00014-83-K-0215
Project NR 047624

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Abstract
of
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SOLUTION AND SENSITIVITY ANALYSIS OF RESDYN MODEL 1a USING SENSUMT

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1. Introduction

Presented in this paper are the results obtained by using SENSUMT to solve "Model 1a," a resource allocation model described in [1] and [2]. Additional background can be obtained from [3] and [4], and a general classification scheme and model designation are given in [5]. Model 1a denotes the model that considers only maintenance allocation. This work differs from that reported in [2], in that SENSUMT is used instead of LINDO, and the original objective function is used rather than a piecewise linear approximation. The use of SENSUMT also allows sensitivity results to be obtained. An extension of Model 1a, incorporating manpower allocation, is reported in [6] but not pursued here.

2. Statement of the Problem

Using the notation and simplified equations from [1] and the definition of the objective function given in [2], the problem can be formulated as a standard nonlinear program:

$$\underset{(x,a,m)}{\text{maximize}} \quad f(x,a,m,\alpha_0) = (1,2,1,1,2) \begin{bmatrix} (a_1 + \alpha_0 m_1)\ (1 - m_1/a_1)^{\alpha_0} \\ (a_2 + \alpha_0 m_2)\ (1 - m_2/a_2)^{\alpha_0} \\ (a_3 + \alpha_0 m_3)\ (1 - m_3/a_3)^{\alpha_0} \\ (a_4 + \alpha_0 m_4)\ (1 - m_4/a_4)^{\alpha_0} \\ (a_5 + \alpha_0 m_5)\ (1 - m_5/a_5)^{\alpha_0} \end{bmatrix}$$

subject to:

(1) $-\alpha_3 a_{t-1} + a_t + x_t = B_0(1 + \alpha_1)^t \ ,$ $\qquad t = 1,2,\ldots,5$

(2) $-\alpha_4 a_{t-1} - [\alpha_3^{\alpha_2} - \alpha_4\alpha_7]\ m_{t-1} + m_t + x_t = 0 \ ,$ $\qquad t = 1,2,\ldots,5$

(3) $-\alpha_4 a_{t-1} - [1 - \alpha_4\alpha_7]\ m_{t-1} + x_t \leqq 0 \ ,$ $\qquad t = 1,2,\ldots,5$

The variables x_t , m_t and a_t denote the maintenance supplied, the maintenance backlog, and the asset value at time t , respectively. The model parameters are as follows:

$\alpha_1 = 0.03$ (yearly proportional budget growth)

$\alpha_2 = 2$ (ratio of maintenance backlog on retiring ships to maintenance average)

$\alpha_3 = 29/30$ (fraction of assets remaining each year)

$\alpha_4 = 0.04$ (fraction of assets requiring annual maintenance)

$\alpha_7 = 0.50$ (fraction of backlog not requiring additional maintenance)

$\alpha_0 = 58$ (Beta-function parameter used in the objective function) .

The initial conditions are as follows:

$B_0 = 8$ (budget)

$a_0 = 100$ (asset value)

$m_0 = 1$ (maintenance)

3. Elimination of the Inequality Constraints

Subtracting Equation (2) from Inequality (3) yields

$$[\alpha_3^{\alpha_2} - 1]\, m_{t-1} - m_t \leqq 0 \,, \quad t = 1,2,\ldots,5 \,.$$

From the definition of these parameters, $\alpha_3 \leqq 1$ and $\alpha_2 \geqq 0$. Therefore, $[\alpha_3^{\alpha_2} - 1] \leqq 0$. Since $m_t \geqq 0$ and $m_{t-1} \geqq 0$, Inequality (3) is redundant and can be eliminated.

4. Attempt at an Analytic Calculation of Solution of RESDYN Model 1a

For a time frame of 5 years, the problem has 15 variables (m_t, a_t, and x_t, $t = 1,2,\ldots,5$) and 10 equality constraints (plus the nonnegativity constraints). It is theoretically possible to use the equalities to eliminate 10 variables, and to formulate the problem in terms of 5 variables, subject to the 15 nonnegativity constraints. This procedure was attempted but proved analytically intractable, due to the complicated expressions that resulted. Therefore, another approach was attempted.

From the results of [2], it is concluded that $m_t = 0$, $t = 1,2,\ldots,5$ in the given solution. Assuming this, we may solve for

x_t and a_t , $t = 1,2,\ldots,5$, and check whether the Karush-Kuhn-Tucker optimality conditions hold at the resulting point and hence whether this point is a candidate solution of the problem as formulated above. (These conditions are necessary at a solution point since the constraints are linear.) If these conditions hold, then we may conclude that the problem may have been solved by the given point, for the given set of parameter values. Solving Equations (1) and (2), under the assumption that $m_t = 0$, $t = 1,2,\ldots 5$, yields the following. For notational convenience, let $[\alpha_3^{\alpha_2} - \alpha_4\alpha_7] = k$.

For $t = 1,\ldots,5$, Equations (1) and (2) imply:

$$
\begin{aligned}
x_1 &= k\, m_0 + \alpha_4 a_0 &&= 4.91444 \\
a_1 &= B_0(1 + \alpha_1) + \alpha_3 a_0 - x_1 &&= 99.99223 \\
x_2 &= \alpha_4 a_1 &&= 3.99969 \\
a_2 &= B_0(1 + \alpha_1)^2 + \alpha_3 a_1 - x_2 &&= 101.14667 \\
x_3 &= \alpha_4 a_2 &&= 4.04587 \\
a_3 &= B_0(1 + \alpha_1)^3 + \alpha_3 a_2 - x_3 &&= 102.47106 \\
x_4 &= \alpha_4 a_3 &&= 4.09884 \\
a_4 &= B_0(1 + \alpha_1)^4 + \alpha_3 a_3 - x_4 &&= 103.96059 \\
x_5 &= \alpha_4 a_4 &&= 4.15842 \\
a_5 &= B_0(1 + \alpha_1)^5 + \alpha_3 a_4 - x_5 &&= 105.61101
\end{aligned}
$$

Substituting our calculated candidate solution, the value of the objective function is $f(x,a,m,\alpha_0) = (1,2,1,1,2)(a_1,a_2,a_3,a_4,a_5)^T = 719.939$.

It is interesting to compare our calculated solution to the solution given in [2], both of which are summarized in the last two columns of Table 1. With the exception of a_5 and the value of the objective function, the results are very close. The objective function value of [2] is higher. We find that the discrepancy in the objective function is due mainly to the different values obtained for a_5. Substituting the variable values of [2] ($x_5 = 4.158567$, $a_4 = 103.964188$, and $a_5 = 109.772942$) into Equation (1) for $t = 5$, we find that the solution of [2] violates this equality constraint. Hence, a_5 as given in [2] is apparently incorrect, and the solution point obtained in [2] is not feasible.

To check our calculated solution candidate, the Lagrangian of the problem is formulated. A change in notation is introduced at this point, for simplicity: $a_1 \equiv x_6$, $a_2 \equiv x_7, \ldots, a_5 \equiv x_{10}$, $m_1 \equiv x_{11}$, $m_2 \equiv x_{12}, \ldots, m_5 \equiv x_{15}$. Also, since we are maximizing the objective function, the negative of the objective function is used in the Lagrangian and in SENSUMT, and the problem is treated as one of minimization. Since, by assumption, only $m_t = 0$, $t = 1,2,\ldots,5$, only the nonnegativity constraints associated with these variables are included.

TABLE 1

PROBLEM INITIAL CANDIDATE SOLUTIONS

	Hand-Calculated Numerical Solution	SENSUMT	§TM-69340/83 (Reference [2])	Calculated with Assumption m = 0
$f^{\dagger}$	720.050	720.050	728.280	719.939
x_1	4.88272	4.88115	4.914398	4.91444
x_2	3.99920	3.99936	3.999825	3.99969
x_3	4.04369	4.04303	4.046005	4.04587
x_4	4.09731	4.09716	4.098984	4.09884
x_5	4.15730	4.15695	4.158569	4.15842
$a_1 = x_6$	100.02395	100.025	99.996	99.992
$a_2 = x_1$	101.17781	101.179	101.150	101.147
$a_3 = x_8$	102.50335	102.505	102.475	102.471
$a_4 = x_9$	103.99334	103.995	103.964	103.961
$a_5 = x_{10}$	105.64379	105.646	109.773†	105.611
$m_1 = x_{11}$	.03172	.0332867	0	0
$m_2 = x_{12}$	.03076	.0320911	0	0
$m_3 = x_{13}$	.03155	.0334766	0	0
$m_4 = x_{14}$	.03168	.0336642	0	0
$m_5 = x_{15}$	.03140	.0336430	0	0
w_1	5.98348	4.42901		
w_2	5.37768	7.53408		
w_3	3.64464	-3.56954		
w_4	2.85376	.044104		
w_5	2.00030	5.92848		
w_6	-5.98348	-9.07268		
w_7	-5.37768	-6.39773		
w_8	-3.64464	-.802994		
w_9	-2.85376	8.71874		
w_{10}	-2.00030	-6.74342		

†See Section 4 above concerning the value of a_5 .

§Point obtained using a piecewise linear approximation of objective function. With a_5 corrected (see Section 4), this point is probably optimal for this approximation. It is not optimal for the original problem.

The resulting Lagrangian is

$$L\,(x,u,w,\varepsilon) = (-1,-2,-1,-1,-2)\begin{bmatrix}(x_6 + \alpha_0 x_{11})(1 - x_{11}/x_6)^{\alpha_0} \\ (x_7 + \alpha_0 x_{12})(1 - x_{12}/x_7)^{\alpha_0} \\ (x_8 + \alpha_0 x_{13})(1 - x_{13}/x_8)^{\alpha_0} \\ (x_9 + \alpha_0 x_{14})(1 - x_{14}/x_9)^{\alpha_0} \\ (x_{10} + \alpha_0 x_{15})(1 - x_{15}/x_{10})^{\alpha_0}\end{bmatrix}$$

$$- u_1 x_{11} - u_2 x_{12} - u_3 x_{13} - u_4 x_{14} - u_5 x_{15}$$

$$\begin{aligned}
&+ w_1\ [x_1 + x_6 - B_0(1 + \alpha_1) - \alpha_3 a_0] \\
&+ w_2\ [x_2 - \alpha_3 x_6 + x_7 - B_0(1 + \alpha_1)^2] \\
&+ w_3\ [x_3 - \alpha_3 x_7 + x_8 - B_0(1 + \alpha_1)^2] \\
&+ w_4\ [x_4 - \alpha_3 x_8 + x_9 - B_0(1 + \alpha_1)^3] \\
&+ w_5\ [x_5 - \alpha_3 x_9 + x_{10} - B_0(1 + \alpha_1)^5] \\
&+ w_6\ [x_1 + x_{11} - \alpha_4 a_0 - k m_0] \\
&+ w_7\ [x_2 - \alpha_4 x_6 - k x_{11} + x_{12}] \\
&+ w_8\ [x_3 - \alpha_4 x_7 - k x_{12} + x_{13}] \\
&+ w_9\ [x_4 - \alpha_4 x_8 - k x_{13} + x_{14}] \\
&+ w_{10}\ [x_5 - \alpha_4 x_9 - k x_{14} + x_{15}]
\end{aligned}$$

where $\varepsilon = (\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_7, B_0, m_0, a_0, \alpha_0)^T$

Taking the gradient of the Lagrangian with respect to the 15 variables gives

$$\nabla_x L = \left[\begin{array}{ll} & \quad 0 \\ & \quad \vdots \\ & \quad 0 \\ \hline (-1) & \left[1 + (\alpha_0-1)\ (x_{11}/x_6) + \alpha_0^2\ (x_{11}/x_6)^2\right](1-(x_{11}/x_6))^{\alpha_0-1} \\ (-2) & \quad \cdot \\ (-1) & \quad \cdot \\ (-1) & \quad \cdot \\ (-2) & \left[1 + (\alpha_0-1)\ (x_{15}/x_{10}) + \alpha_0^2\ (x_{15}/x_{10})^2\right]\ (1-(x_{15}/x_{10})^{\alpha_0-1} \\ \hline (+1) & \alpha_0(\alpha_0+1)\ (x_{11}/x_6)\ (1-(x_{11}/x_6)^{\alpha_0-1} \\ (+2) & \quad \cdot \\ (+1) & \quad \cdot \\ (+1) & \quad \cdot \\ (+2) & \alpha_0(\alpha_0+1)\ (x_{15}/x_{10})\ (1-(x_{15}/x_{10})^{\alpha_0-1} \end{array}\right]$$

$$- \left[\begin{array}{c} 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \\ \hline 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \\ \hline u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{array}\right] + \left[\begin{array}{l} w_1 + w_6 \\ w_2 + w_7 \\ w_3 + w_8 \\ w_4 + w_9 \\ w_5 + w_{10} \\ \hline w_1 - \alpha_3 w_2 - \alpha_4 w_7 \\ w_2 - \alpha_3 w_3 - \alpha_4 w_8 \\ w_3 - \alpha_3 w_4 - \alpha_4 w_9 \\ w_4 - \alpha_3 w_5 - \alpha_4 w_{10} \\ w_5 \\ \hline w_6 \ - k\ w_7 \\ w_7 \ - k\ w_8 \\ w_8 \ - k\ w_9 \\ w_9 \ - k\ w_{10} \\ w_{10} \end{array}\right]$$

Let $x_{11} = x_{12} = x_{13} = x_{14} = x_{15} = 0$. Then

$$
= \left[\begin{array}{c} 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \\ \hline -1 \\ -2 \\ -1 \\ -1 \\ -2 \\ \hline 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{array}\right] - \left[\begin{array}{c} 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \\ \hline 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \\ \hline u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{array}\right] + \left[\begin{array}{l} w_1 + w_6 \\ w_2 + w_7 \\ w_3 + w_8 \\ w_4 + w_3 \\ w_5 + w_{10} \\ \hline w_1 - \alpha_3 w_2 - \alpha_4 w_7 \\ w_2 - \alpha_3 w_3 - \alpha_4 w_8 \\ w_3 - \alpha_3 w_4 - \alpha_4 w_9 \\ w_4 - \alpha_3 w_5 - \alpha_4 w_{10} \\ w_5 \\ \hline w_6 - k\,w_7 \\ w_7 - k\,w_8 \\ w_8 - k\,w_9 \\ w_9 - k\,w_{10} \\ w_{10} \end{array}\right]
$$

The gradient of the Lagrangian is set equal to zero to see if the Karush-Kuhn-Tucker first order necessary conditions hold. From row 10, $w_5 = 2$. From row 5, $w_{10} = -2$. From row 15, $u_5 = -2$. Since the multipliers associated with inequality constraints must be nonnegative for the point to be a minimizer (maximizer of the original problem), the calculated candidate solution point is not even a local minimum. Hence, the variables $m_1, \ldots, m_5$ cannot be equal to 0 at a solution of RESDYN Model 1a.

It should be noted in passing that [2] was obtained by using a piecewise linear approximation of the objective function. If the gradient of the Lagrangian is reformulated using this approximation,

it may be shown that the first order necessary conditions are satisfied at our calculated solution point.

Returning to our analysis of the original problem, since none of the nonnegativity constraints are binding, it follows that all the $u_i = 0$. We now find that the gradient of the Lagrangian can be set equal to zero and that we can solve analytically for the multipliers w_j and the ratios $x_{11}/x_6, x_{12}/x_7, \ldots, x_{15}/x_{10}$. For notational convenience, define $z_1 = x_{11}/x_6,\ z_2 = x_{12}/x_7, \ldots,\ z_5 = x_{15}/x_{10}$.

From row 5: $w_5 = -w_{10}$

From row 10: $(-2)\left[1 + (\alpha_0-1)z_5 + \alpha_0^2 z_5^2\right](1-z_5)^{\alpha_0-1} + w_5 = 0$

From row 15: $(+2)\ \alpha_0(1+\alpha_0)z_5(1-z_5)^{\alpha_0-1} + w_{10} = 0$

Therefore, $\left[1 + (\alpha_0-1)z_5 + \alpha_0^2 z_5^2\right](1-z_5)^{\alpha_0-1} = \alpha_0(1+\alpha_0)z_5(1-z_5)^{\alpha_0-1}$,

so $z_5 = 1$ or $1 + (\alpha_0-1)z_5 + \alpha_0^2 z_5^2 - \alpha_0(1+\alpha_0)z_5 = 0$

yielding $\alpha_0^2 z_5^2 - (\alpha_0^2+1)z_5 + 1 = 0$.

Using the formula for the solution of a quadratic equation yields

$$z_5 = \frac{(\alpha_0^2+1) \pm (\alpha_0^2-1)}{2\alpha_0^2} ,$$

and hence $z_5 = 1$ and $w_5 = w_{10} = 0$ or $z_5 = 1/\alpha_0^2$. It may be argued that the smaller value of z_i will reduce f , hence $z_5 = 1$ is discarded. Substituting back into either row 10 or 15 with $z_5 = 1/\alpha_0^2$ yields

$$w_5 = -w_{10} = 2(1+1/\alpha_0)(1-1/\alpha_0^2)^{\alpha_0-1} .$$

Since $w_4 = -w_9$ from row 4, the values for w_5 and w_{10} can be substituted into rows 9 and 14 to solve for z_4 . However, these equations

cannot be solved in closed form, and in particular one cannot as before divide by $(1-z_4)^{\alpha_0-1}$ to obtain a quadratic equation. The equations were solved numerically, using Newton's method, substituting the nominal values for the parameters. Substituting the calculated values of the multipliers into the other rows (8 and 13, then 7 and 12, etc.), all the ratios and multipliers were solved by this numerical method. Using the numerical values for $z_1, z_2, \ldots, z_5$ and fixing the parameters at thier nominal values, the recursive equations (1) and (2) of Section 2 were solved to obtain a candidate solution point. The optimal value, candidate solution point. The optimal value, candidate solution point and multipliers resulting from this method are given in Table 1.

5. Solution using SENSUMT

The problem was solved using SENSUMT running on a DEC VAX 11/780. The method selected for minimizing the unconstrained penalty function was G. P. McCormick's modification of the Fletcher-Powell method (i.e., NEXOP2 = 4) . The results are summarized in Table 1. For comparison, and although they are not feasible, the results of [2] and the results that follow from the closed form solution of the equations with the assumption that $m_t = 0$, $t = 1,2,\ldots,5$, are also presented, as noted earlier.

The solution point and estimates of the multipliers were compared to the results of the method described above in Section 4 (forming the gradient of the Lagrangian with respect to the 15 variables and setting this to zero).

The SENSUMT estimates of the multipliers do not appear to be accurate. These estimates are obtained from the ratio $w_j = 2h_j/\rho$, where $\rho > 0$ is the penalty function algorithm parameter set by the user. Since both the constraint value h_j and ρ go to zero as the program iterates, the resulting ratio of two small numbers may not be accurate. The accuracy of the solution point is much better than that of the multipliers. All the variables agree to the third place after the decimal point. The optimal value (720.050) is in agreement to the accuracy kept in the numerical solution.

6. Sensitivity Analysis

The use of SENSUMT allows sensitivity results to be obtained. The sensitivity of the optimal value, solution point and multipliers were obtained for nine parameters: $\varepsilon = (\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_7, B_0, a_0, m_0, \alpha_0)^T$, where α_0 is the parameter in the objective function. The gradients of the optimal value function f^*, the solution point, and Lagrange multipliers are given in Tables 2, 3 and 4, respectively. Note that for the sake of the sensitivity analysis, the initial conditions $(B_0, a_0$ and $m_0)$ are treated as parameters.

The above gradients represent a rate of change in the variable with respect to each parameter. These gradient values are valid only at the nominal parameter values, and can change significantly as a parameter is varied from its nominal value. Also, some parameters can vary over a much wider range than others. For example, a parameter such as a_0 (initial asset value), with a nominal value of 100, can change over a larger range than α_1 (proportional budget growth), which has a nominal value of 0.03 .

TABLE 2

SENSITIVITY OF $f^*(\varepsilon)$

$$\nabla_{\varepsilon}^{T} f^{*} = \begin{bmatrix} 419.141 \\ -10.1688 \\ 2052.00 \\ -1836.59 \\ -12.8398 \\ 21.3820 \\ 5.54479 \\ -5.47166 \\ 26.5243 \end{bmatrix} \quad \text{where} \quad \varepsilon = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \\ \alpha_7 \\ B_0 \\ a_0 \\ m_0 \\ \alpha_0 \end{bmatrix}$$

TABLE 3

SENSITIVITY OF SOLUTION POINT

$$\nabla_{\varepsilon}^{T}x = \begin{bmatrix} 2.2E-5 & .3200 & .9558 & 1.904 & 3.163 \\ -235.5 & 197.3 & 350.8 & -563.4 & 242.2 \\ 5597. & 120.5 & -25145. & 27484. & -8041. \\ 3795. & -3030. & -5316. & 8891. & -3733. \\ -297.3 & 249.1 & 443.0 & -711.4 & 305.8 \\ 3.0E-6 & 4.1E-2 & 8.1E-2 & .1184 & .1547 \\ 4.0E-2 & 3.7E-2 & 3.4E-2 & 3.2E-2 & 2.9E-2 \\ .9144 & -3.7E-2 & -3.4E-2 & -3.1E-2 & -2.9E-2 \\ 43.95 & 36.02 & -30.78 & 2.044 & 44.18 \end{bmatrix}$$

$$\nabla_{t}^{T}a = \begin{bmatrix} 8.000 & 23.89 & 47.60 & 79.08 & 118.3 \\ 235.5 & 30.37 & -321.5 & 252.6 & 2.048 \\ -5497. & -5334. & 20090. & -7962. & 448.0 \\ -3795. & -638.2 & 4699. & -4349. & -471.2 \\ 297.3 & 38.35 & -405.9 & 319.0 & 2.586 \\ 1.030 & 2.015 & 2.960 & 3.869 & 4.744 \\ .9267 & .8587 & .7957 & .7374 & .6833 \\ -.9144 & -.8474 & -.7852 & -.7277 & -.6743 \\ -43.95 & 78.50 & -45.10 & -45.64 & -88.30 \end{bmatrix}$$

$$\nabla_{t}^{T}m = \begin{bmatrix} -2.2E-5 & -3.9E-5 & -5.9E-5 & -7.2E-5 & 2.0E-4 \\ 235.5 & 27.46 & -324.5 & 253.8 & 1.6E-2 \\ -5595. & -5456. & 19942. & -8445. & -.5265 \\ -3695. & -400.8 & 5025. & -4006. & -.2443 \\ 297.3 & 34.67 & -409.7 & 320.5 & 2.0E-2 \\ -3.0E-6 & -5.4E-6 & -2.4E-6 & -5.1E-6 & 9.7E-6 \\ -7.1E-7 & 2.1E-7 & 2.2E-7 & 2.8E-7 & 1.9E-6 \\ 4.5E-5 & -2.0E-7 & -2.1E-7 & -2.8E-7 & -1.8E-6 \\ -43.95 & -77.96 & -43.65 & -43.76 & -86.02 \end{bmatrix}$$

Recall that $a_1 = x_6, \ldots, a_5 = x_{10}, \; m_1 = x_{11}, \ldots, m_5 = x_{15}$.

TABLE 4

SENSITIVITY OF LAGRANGE MULTIPLIERS

$$\nabla_{\varepsilon}^{T} w^{1} = \begin{bmatrix} -2.7E-8 & -2.8E-8 & -2.6E-8 & -1.9E-8 & -1.1E-8 \\ -5.4E-7 & -6.0E-8 & 8.2E-7 & -5.4E-7 & -1.9E-9 \\ 8.982 & 1.562 & 5.538 & 5.928 & 1.2E-7 \\ -5.021 & 1.486 & 2.470 & -6.743 & 1.6E-7 \\ -7.0E-7 & -8.2E-8 & 1.0E-6 & -6.6E-7 & 3.4E-9 \\ -6.6E-9 & -6.8E-9 & -3.8E-9 & -3.6E-9 & -2.9E-10 \\ -5.8E-10 & -4.4E-10 & -2.9E-10 & -1.7E-10 & -5.8E-11 \\ 5.8E-10 & 4.4E-10 & 2.9E-10 & 1.5E-10 & 2.9E-11 \\ 3.5E-5 & 3.1E-5 & 2.1E-5 & 1.6E-5 & 1.1E-5 \end{bmatrix}$$

$$\nabla_{\varepsilon}^{T} w^{2} = \begin{bmatrix} 2.8E-8 & 9.4E-9 & -3.0E-8 & -8.7E-8 & -1.7E-7 \\ 1.0E-5 & 1.2E-5 & -2.1E-5 & 3.2E-5 & -1.3E-5 \\ -8.982 & -1.562 & -5.536 & -5.930 & 4.4E-4 \\ 5.021 & -1.486 & -2.470 & 6.743 & 2.1E-4 \\ 1.3E-5 & -1.5E-5 & -2.7E-5 & 4.1E-5 & -1.7E-5 \\ 6.6E-9 & 4.4E-9 & -7.0E-10 & -3.0E-9 & -8.2E-9 \\ -9.9E-10 & -1.7E-9 & -1.7E-9 & -1.6E-9 & -1.6E-9 \\ -3.7E-8 & 1.7E-9 & 1.7E-9 & 1.6E-9 & 1.6E-9 \\ -3.7E-5 & -3.3E-5 & -1.9E-5 & -1.7E-5 & -1.4E-5 \end{bmatrix}$$

$w^{1} = (w_1, \ldots, w_5)^T$

$w^{2} = (w_6, \ldots, w_{10})^T$

In order to identify those parameters that have the greatest impact on the solution of the problem, it is useful to calculate for each parameter the change in the optimal value resulting from a change of a fixed proportion of the range associated with the parameter. An estimate of the range for each parameter was obtained from [7]. These ranges are given in Table 5. For each parameter, an estimate of the change in the optimal value due to an increase in the parameter of 1% of the range (while all other parameters are held at their nominal values) is calculated. This calculation is based on an extrapolation of the first order sensitivity estimates at the solution, i.e., $\nabla_\varepsilon f^*$. Since this extrapolation is linear, the change in the optimal value function is proportional to the proportion of the range selected. Although the proportion chosen is 1%, the extrapolation could be performed for any proportion. These changes in f^* due to each parameter are given in Table 6.

It is useful to rank the parameters in order of their impact on the optimal value, using the absolute value of the elements of the gradient and the changes of the optimal value due to a change of a fixed proportion of the range of each parameter. For the gradient, the order of sensitivity to the parameters (from greatest to least sensitivity) is α_3, α_4, α_1, α_0, B_0, α_7, α_2, a_0, and m_0. Notice the dominance of α_3 and α_4 on the optimal value, and also on the solution point and multipliers (from Tables 3 and 4). To a lesser extent,

TABLE 5

RANGES ASSOCIATED WITH THE PARAMETERS

Parameter	Nominal Value	Lower Bound	Upper Bound	Range
α_1	0.03	-0.05	0.15	0.20
α_2	2.0	1.0	4.0	3.0
α_3	0.967	0.75	1.0	0.25
α_4	0.04	0.01	0.10	0.09
α_7	0.50	0	1.0	1.0
B_0	8	5	20	15
a_0	100	80	300	220
m_0	1	1	10	9
α_0	58	5	60	55

TABLE 6

CHANGES IN $f^*(\varepsilon)$ FOR AN INCREASE OF 1% OF THE RANGE FOR EACH PARAMETER

Parameter	$\nabla_\varepsilon f^*(\varepsilon)$	Δf^*	$f^* + \Delta f^*$
α_1	419.141	.8383	720.8883
α_2	-10.1688	-.3051	719.7449
α_3	2052.00	5.1300	725.1800
α_4	-1836.59	-1.6529	718.3971
α_7	-12.8398	-.1284	719.9216
B_0	21.3820	3.2073	723.2573
a_0	5.54479	12.1985	732.2485
m_0	-5.47166	-.4924	719.5576
α_0	26.5243	14.5884	734.6384

the optimal value is sensitive to α_1 , while the solution point is sensitive to α_2 and α_7 . All other sensitivities are quite low. For the changes in the optimal value due to a range proportional change in each parameter, the order of sensitivity is α_0, a_0, α_3, B_0, α_4 α_1 , m_0, α_2, and α_7 . Notice how the ranking changes due to the variance in the ranges. Now α_0 and a_0 are the most critical parameters in the optimal value.

In order to get an estimate of lower and upper bounds on the optimal value, the SENSUMT program was run with the parameters set to their extreme values. The particular extreme value (upper or lower bound) depended on the sign of the corresponding element in the gradient. The estimate for a lower bound on $f^*(\varepsilon)$ is 248.844. However at this solution point the sign of the gradient changes for α_2, α_3, and α_7 compared to the gradient at the nominal solution point. Therefore it is possible to obtain an even lower optimal value at some intermediate values of these parameters. The estimate of the upper bound is 2647.162. All elements of the gradient have the same sign as the gradient at the nominal solution. Therefore, it is plausible that this estimate of the upper bound is valid.

Calculation of parametric optimal value bounds for this nonconvex program and analyses for extensions of this model will be a subject of future research.

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